System Size Identification from Sinusoidal Probing

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One of the most fundamental characteristic of a complex system is its size (or volume), which, in many modelling, is represented by the number of its individual components. Complex systems under investigation nowadays are typically large and/or time-varying, rendering their identification challenging. We propose here an accurate and efficient method to determine the size of (i.e., number of agents in) a complex dynamical system. Our only requirement is to be able to inject a probing signal at any point of the system and to measure its response to our probing. The complexity of the approach depends only on the length of the measurements and not on the size of the system investigated, which renders it extremely efficient.

Introduction. At the era of big data, access to high-resolution (in space and time) measurements is increasingly easy. In parallel, the ever-improving computational power of computers allows for the analysis of such large sets of data [1]. Among many examples, these considerations apply to domains ranging from electrical grid to social networks and gene regulatory networks. Phasor measurement units [2] are broadly installed in the electrical grid and provide high-frequency measurements of voltage and current over large interconnected areas of the grid. Similarly, online social networks gather a significant fraction of the world population with the ability to monitor their opinion in real time [3]. And modern biochemical technologies allow to assess the expression of a large variety of genes governed by gene regulatory networks [4]. Overall, it is a general trend that an increasing number of data are available covering larger and larger physical and virtual systems. With machine learning leading the way, modern technologies promise to leverage this increasing amount of data to improve the wellbeing of humanity in the future [5].

However, the other side of the coin of this tremendous amount of data is that it comes with significant uncertainties. Indeed, it is technically impossible to permanently monitor the monitoring system itself. Some measurements might be inaccurate, wrong, or even missing. Furthermore, in general, it is not possible to guarantee that each component of a large system will be monitored at all time, either due to the number of units to be monitored or to the time-varying nature of the system’s components. These unavoidable inaccuracies and uncertainties can jeopardize the efficacy of data-based technologies.

In this scope, it is of particular interest to be able to recover the system’s characteristics (parameters, internal structure, etc.) from the measured data. The more efficient such inference method are, the closer to real-time it can be performed, allowing an accurate picture of the system at all time. For instance, recovering the underlying structure of a network of dynamical agents, based on measurements, has been an active topic of research along the last decades [6–13]. However, the majority of those approaches rely on the knowledge of most (if not all) the agents composing the system. Sometimes, a subset of these agents is not accessible to the observer, rendering the recovery of the network harder, if not impossible [14]. In the worst situations, it even happens that the observer does not even know the actual size of the system [15].

The number of components in a system is one of its most fundamental characteristics and is often unknown, especially for large, time-varying systems. A crucial step in the process of system identification is then to recover it as efficiently as possible, i.e., as accurately and with as few measurements as possible. Despite its apparent simplicity, this problem is actually not trivial and surprisingly underinvestigated given its fundamental relevance in the scope of system identification. The authors of [14] proposed an approach to locate hidden nodes in a networked dynamical system, which relies on the comparison between measured and predicted trajectories of the accessible agents of the system. This method requires a good knowledge of the differential equations determining the dynamics of the system in order to numerically integrate them. Moreover, the authors assume that the number of unaccessible agents is very small (usually only one).

As far as we can tell, the most up-to-date approach to recover the number of agents in a system has been detailed in [15]. This approach recovers the number of units as the rank of a detection matrix constructed with time series of the measured units. Namely, it requires to observe the system at $k$ time steps, along $M$ different trajectories, and if $k$ and $M$ are large enough, in principle, the total number of dynamical units can be recovered. In summary, if $N$ is the number of observable units, this method relies on $NkM$ measured values and requires to be able to set the system in $M$ different initial conditions. A recent Letter [16] elegantly draws the link between the detection matrix of [15] and the observability matrix commonly used in control theory [17]. Ref. [16] shows that the approach proposed in [15] can unambiguously determine the system size if and only if the system
is observable in the control-theoretic sense.

We show here that an accurate determination of the number of units composing a physical system can proceed via measurement of the trajectory of one or two dynamical units, which reduces significantly the number of measurements compared to the state-of-the-art literature [15]. The cost of our approach is that we require to be able to inject a probing signal at one of the nodes of the network. In counterpart, we are able to accurately recover the number of agents by injecting a probing signal at unit $i$ and by measuring the induced response at unit $j$ (possibly equal to $i$). Moreover, the numerical complexity of our method is independent of the system size, rendering it extremely scalable.

We argue that our method can hardly be more efficient. Indeed, extracting information from a system requires to be able to measure some output (we require one time series) and to have some information about the inputs that triggered the measured response (we require a controlled probing signal).

**System of coupled units.** Let us consider a general system of $n$ coupled dynamical units with first-order dynamics

$$
\dot{x}_i = \omega_i - \sum_{j=1}^{n} a_{ij} f_{ij}(x_i - x_j) + \xi_i, \quad i = 1, \ldots, n,
$$

where $x_i \in M$ is the time-varying value of the $i$th agent, evolving on a one-dimensional manifold $M$, $\omega_i \in \mathbb{R}$ is the natural driving term of agent $i$, and $\xi_i$ will be used as an input to the system. Two agents $i$ and $j$ are interacting if a link between them exists in the interaction network, i.e., if and only if the corresponding term of the adjacency matrix $a_{ij} = 1$. We assume that the interaction graph is connected (otherwise we restrict ourselves to a connected component). The interaction function between $i$ and $j$ is an odd, differentiable function $f_{ij} : \mathbb{R} \to \mathbb{R}$, and we consider the coupling to be attractive ($\partial f_{ij}/\partial x > 0$) in an interval around $x = 0$.

If a fixed point $x^{*} \in M^n$ exists, one can linearize Eq. (1) around it, which yields, for a small deviation $\delta x = x - x^{*}$, to the approximate dynamics

$$
\dot{\delta}x = -J(x^{*})\delta x + \xi,
$$

where we use the Jacobian matrix of Eq. (1),

$$
J_{ij}(x^{*}) = \begin{cases} 
-a_{ij} \partial_x f_{ij}(x) \big|_{x=x^*_i-x^*_j}, & \text{if } i \neq j, \\
\sum_{k \neq j} \partial_x f_{ik}(x) \big|_{x=x^*_i-x^*_k}, & \text{if } i = j.
\end{cases}
$$

One can verify that the structure of the interaction implies that the Jacobian $J$ is a weighted directed Laplacian matrix of the interaction graph. Let us denote its right- (resp. left-) eigenvectors $u_1, \ldots, u_n$ (resp. $v_1, \ldots, v_n$). From now on, we will focus on stable fixed points of Eq. (1), implying that the eigenvalues have non-negative real part, $0 = \lambda_1 < \text{Re}(\lambda_2) \leq \ldots \leq \text{Re}(\lambda_n)$ (note that one eigenvalue is always zero) and the eigenvectors form a basis of $\mathbb{R}^n$.

Equation (2) is then solved by expanding the deviation $\delta x$ over the eigenvectors $v_\alpha$ of $J$, i.e., $\delta x_i(t) = \sum_{\alpha} c_\alpha(t) v_{\alpha,i}$, yielding a set of Langevin equations in $c_\alpha$, whose solution is

$$
c_\alpha(t) = e^{-\lambda_\alpha t} \int_0^t e^{\lambda_\alpha t'} v_{\alpha,i} \xi(t')dt',
$$

for $\alpha = 1, \ldots, n$. A detailed derivation of this result can be found in [18], we do not reproduce it here for sake of brevity.

**Sinusoidal probing.** We will say that a pair of agents $(i,j)$ can be probed if we have the ability to inject a probing signal at one of them and to measure the response at the other. Note that we accept that $i = j$.

We propose now to inject a sinusoidal signal at agent $i$ and to measure its impact at agent $j$. Let

$$
\xi_i(t) = a_0 \sin(\omega_0 t),
$$

be the probing signal at agent $i$. We do not inject a probing signal at other nodes. To guarantee a minimal impact on the operation of the system, we keep the amplitude $a_0$ and most importantly the frequency $\omega_0$ to small values. Keeping a small probing frequency guarantees that the system can adapt to the input and follow the probing signal. More precisely, a probing frequency can be qualified as "small" as long as it is smaller than the smallest (nonzero) eigenvalue of the Jacobian matrix $J$ in absolute value.

Introducing Eq. (5) into Eq. (4), and recombining the eigenmodes yields the following response measured at agent $j$ while probing agent $i$,

$$
x_j^i(t) = \sum_{\alpha} \frac{v_{\alpha,i} u_{\alpha,j} a_0}{\lambda_\alpha^2 + \omega_0^2} \times \left[ \lambda_\alpha \sin(\omega_0 t) + \omega_0 e^{-\lambda_\alpha t} - \omega_0 \cos(\omega_0 t) \right],
$$

which, in the long time limit $\lambda_\alpha t \gg 1$ and with the asymptotic $\omega_0 \ll \lambda_\alpha$, yields

$$
x_j^i(t) = \int_{j}^{i} a_0 \sin(\omega_0 t) + \frac{a_0}{n \omega_0} \left[ 1 - \cos(\omega_0 t) \right],
$$

where the $\dagger$ denotes the Moore-Penrose pseudo-inverse.

From here, we distinguish two cases. First, if it is possible to probe the system with a signal frequency which is much smaller than all characteristic times of the systems, i.e., $\omega_0 \ll \lambda_n$ for all $\alpha = 2, \ldots, n$, then the first term in the right hand side of Eq. (7) can be neglected with respect to the second term. Considering the maximal value of the trajectory of $x_j$, one then gets

$$
\max_{t} |x_j^i(t)| \approx \frac{2a_0}{n \omega_0},
$$
from which one obtains an estimate for the number of nodes as,

\[ \hat{n} = \frac{2a_0}{\omega_0 \max_i |x_j^r|}. \]  

(9)

The only requirement is that the time series is long enough to reach the maximum of the trajectory.

Note that we take the maximum to have a better accuracy in the estimation. However one can choose a particular time step \( t \), keeping in mind that \( t \) too short leads to vanishing values for Eq. (7).

The second case we consider is when, for some reason, it is not possible to probe with a signal with sufficiently low frequency, meaning that the first term in the right hand side of Eq. (7) cannot be neglected with respect to the second. In this case, we are able to estimate the number of agent in the system provided we can measure the trajectories of two distinct agents \( j \) and \( k \) while we inject the probing signal at agent \( i \) (which, again, can be \( j \) or \( k \)). One realizes that the trajectories \( x_j(t) \) and \( x_k(t) \) differ in the first term of the right hand side of Eq. (7). But this term vanishes for all \( t = k\pi/\omega_0, k \in \mathbb{Z} \). At these time steps, all trajectories then coincide, and take value either zero, or \( \dot{x} := 2a_0/n\omega_0 \). The number of agents can then be estimated as

\[ \hat{n} = \frac{2a_0}{\omega_0 \dot{x}}. \]  

(10)

As previously, we require that the measured trajectories are long enough to reach the intersection.

Note that if, for some reason, \( J_{ij}^f = J_{ik}^f \) (which we cannot exclude a priori), then the two trajectories coincide for all \( t \). In this case, \( n \) cannot be estimated based on these two trajectories (which are technically the same).

Numerical validation. In order to get numerical confirmation of our results, we will apply them to the Kuramoto model on three different interaction graphs. Each system has the same number of units, \( n = 3809 \), but significantly different network structure:

**ER:** An Erdős-Rényi graph with edge probability \( p = 0.003 \) (\( m = 21444 \) in our realization);

**SW:** A Small-World constructed following the Watts-Strogatz process \([19]\), with \( m = 38090 \) edges and rewiring probability \( p = 0.01 \);

**EU:** The more realistic network of the PanTaGruEl model of the European interconnected high voltage grid \([20, 21]\), composed of \( m = 4955 \) edges.

For probing, we use a sinusoidal signal with controlled amplitude and frequency, similarly as what is typically used to identify eigenmodes in electrical networks \([22]\). The relative errors of the estimate of \( n \) for each of the three networks considered and for different probing frequencies is shown in Fig. 1. For small frequency of probing [panels (g-i)], all trajectories almost coincide and we can use Eq. (9) to estimate the number of agents. For larger frequencies [panels (a-f)], the trajectories are distinct, but intersect all at the same time step, and Eq. (10) can be used to estimate the number of nodes. Overall, we see an extremely good accuracy with the trajectories considered, except in Fig. 1(c). We recall that the computation time required by these estimations is independent of the actual size of the system as we only need to work with one or two time series.

Determining what a "small" probing frequency depends on the Jacobian matrix. More precisely, a probing frequency can be considered as "small" if it is sufficiently smaller than all the intrinsic time scales of the system, i.e., smaller than all eigenvalues of the Jacobian. In practice, this means that the intrinsic dynamics of the system relax before the probing signal significantly changes. If the probing frequency is not small enough, the system will not relax before the probing significantly changes. In such a case, our approximation Eq. (7) will be inaccurate and consequently the estimate \( \hat{n} \) as well. This explains why the estimate in Fig. 1(c) is not very accurate. Also, if \( J \) has a lot of small eigenvalues (in the sense that they are close to \( \lambda_2 \)), the addition of all the slow-relaxing modes will take a long time to have a negligible contribution to the trajectories Eq. (6). An accurate estimate of \( \hat{n} \) will then require a smaller probing frequency, and this explains why different \( \omega_0/\lambda_2 \) ratios are considered for different networks in Fig. 1. In Fig. 2 we show the histograms of the eigenvalues of the Jacobian for each system considered in Fig. 1. Unravelling with more precision the relation between the distribution of eigenvalues and the probing frequency will be the purpose of a future work.

**Conclusion.** As far as we can tell, we improved the current state-of-the-art approaches to estimate the size of a complex dynamical system, based on one time series measurement solely. The computation cost is low and independent of the system's size. The only requirements are that we need to be able to probe the system with a signal (typically with low amplitude and frequency) and to measure its response.

In our opinion, the performance of our method (in terms of cost of data acquisition and computation) is close to be as optimal as possible. Indeed, in order to get information about the system, one needs to measure something, i.e., at least one output, which is what we do. Furthermore, to be able to analyze the output of the system, the observer needs some information about the input that triggered the response. Aside of our approach which is to have a direct control on the input signal, one could monitor and analyze a non-controlable input signal \([13]\), but this would require more computation time. A possible improvement could be to rely on shorter time series, which can be long in our case, due to the low frequency of the probing signal.
Figure 1. Histogram of the relative error on the number of nodes obtained for the two methods Eq. (9) and Eq. (10). We consider an Erdős-Rényi graph (first row) with \( m = 21444 \) edges, a Small-World network (second row) with \( m = 38090 \) edges, and rewiring probability \( p = 0.01 \), and the network of the PanTaGruEl model of the European electrical grid (third row) with \( m = 4955 \) edges. All networks have \( n = 3809 \) vertices. We consider three probing frequency regimes, decreasing from left to right panels. Ratios between probing frequencies and \( \lambda_2 \) are shown in insets. Insets of panels (a-f) show all nodes’ trajectory as response to a single node probing, while insets of panels (g-i) show trajectories obtained from probing and measurement at a single node for 39 different network nodes. The estimate of the number of agents in the system is given by the height of the intersection of the curves in the insets of panels (a-f) [see Eq. (10)] and by the maximal value of the curves in panels (g-i) [see Eq. (9)]. Percentages at the top of each panel correspond to the fraction of nodes for which the error was more than 1%, delimited by the vertical dashed lines.

Figure 2. Histograms of the eigenvalues of the three networks considered (normalized by the smallest nonvanishing eigenvalue \( \lambda_2 \)). We observe very different distributions for different network topologies.

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